# Danish Meteorological Institute

Ministry of Transport


## Scientific Report 07-08

# An efficient method to solve implicit generalized similarity equations for the stable and neutral boundary layer


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# Colophone

**Serial title:**
Scientific Report 07-08

**Title:**
An efficient method to solve implicit generalized similarity equations for the stable and neutral boundary layer

**Subtitle:**

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**Other Contributers:**

**Responsible Institution:**
Danish Meteorological Institute

**Language:**
English

**Keywords:**
HIRLAM, stable boundary layer, generalized similarity, truly neutral, conventional stable, long-lived stable

**Url:**
www.dmi.dk/dmi/sr07-08

**ISSN:**
1399-1949

**ISBN:**
978-87-7478-556-9

**Version:**
**Website:**
www.dmi.dk

## Dansk Resume

En generalisering af similaritetsteorien for for det neutrale og stabile atmosfæriske grænselag er for nylig (2007) blevet foreslået af Zilitinkevich og Esau. Til forskel fra den gængse teori afhænger turbulensen i grænselaget også af den statiske stabilitet i luftlaget over grænselagets top. Desuden betragtes de turbulente transporter af impuls, varme og fugtighed ikke længere som uafhængige af højden over overfladen. I den gængse teori antages, at der ved bunden af grænselaget eksisterer et overfladelag, hvor turbulenstransporterne er konstante med højden. De generaliserede similaritetsligninger for impuls, temperatur og fugtighed er implicitte ligninger, som løses iterativt. For at kunne komme i betragtning i en operationael numerisk vejrmodel er kravet, at iterationen konvergerer hurtigt. I nærværende rapport beskrives en løsningsmetode, som har hurtig kongergens i et meget bredt parameterrum, mere specifikt fra neutral til stærk stabil stratificering i og over grænselaget, fra en glat til en ru overflade og fra tropiske til polare grænselag. I dette parameterrum er løsningen efter to iterationer praktisk taget identisk med løsningen efter 50 iterationer. Vinddrejningen med højden op gennem grænselaget er et tilbageværende problem. Rapporten giver et forslag til en mulig løsning på dette problem.

# Abstract

# Introduction

The present report contains a pilot study with the purpose of investigating the feasibility of appying a parameterization of turbulence in the neutral and stable planetary boundary layer (PBL) based on a newly developed generalization of the Monin-Obukhov similarity theory (Zilitinkevich and Esau, 2007). The parameterization as it is presented in Zilitinkevich and Esau, 2007, requires iterative solutions of similarity equations for local turbulence fluxes and thereafter iterative solutions for surface fluxes and PBL height. A key demand for possible implementation in numerical weather prediction models (NWP's) is therefore fast convergence of the iterative solutions in the entire parameter space. It is investigated to what extend fast convergence can be obtain with starting iteration values of local turbulence fluxes obtained by estimates based on the bulk Richardson number. Section 1 contains a brief review of the Monin-Obukhov similarity theory for the horizontally homogeneous atmospheric surface layer. This is followed by a short presentation of a recently developed generalization of the classical similarity theory for the neutral and stably stratified PBL in section 2. In the latter section a possible extension of the equilibrium PBL height equation to the equatorial region based on similarity arguments is outlined. In section 3 an iterative solution method for local turbulence fluxes is described. A similar iterative solution method for surface fluxes and PBL height is described in section 4. Both iteration methods show very fast convergence in the entire parameter space. Finally, section 5 contains discussion and concluding remarks. The discussion mainly deals with the problem concerning how to calculate the angle $\alpha(z)$ between the stress $\vec{\tau}(z)$ at height $z$ and the surface stress $\vec{\tau}_*$. Knowledge of $\alpha(z)$ is needed in the HIRLAM NWP model. A tentative equation for $\alpha(z)$ is suggested.

# Classical Monin-Obukhov similarity

According to the hypothesis of Monin-Obukhov vertical gradients of wind, temperature and moisture in the surface layer only depends on the height $z$ above the surface, the buoyancy parameter $\beta = g\theta_0^{-1}$ and the kinematic turbulent surface fluxes $\tau_*$, $F_{\theta *}$ and $F_{q*}$ of momentum, sensible heat and moisture, respectively. In the buoyancy parameter $g$ is gravity and $\theta_0$ a representative temperature in the surface layer. The sensible heat and moisture flux can be combined into a buoyancy flux $F_{b*} \approx \beta \cdot F_{\theta *} + 0.61 \cdot g \cdot F_{q*}$ by applying the equation for virtual potential temperature. Formally, it then follows from the Buckingham Pi theorem that the relationship between the four dimensional quantities can be reduced to a relationship between two non-dimensional quantities. If we take the velocity gradient as an example the dimensional relationship $\phi_u(\partial U/\partial z, z, \tau_*, F_{b*}) = 0$ is reduced to an equation between two non-dimensional quantities, i.e.

$$\Pi_1 = \Phi_{U0}(\Pi_2) \tag{1}$$

In a neutrally stratified boundary layer $F_{b*} = 0$ and therefore $\phi_u(\partial U/\partial z, z, \tau_*) = 0$ leading to $\Pi_1 = z \cdot \tau_*{}^{-1/2} \partial U/\partial z = k_m$, where $k_m$ is a non-dimensional constant equal to $1/k$, where $k = 0.4$ is the Von Karman constant. In the more general non-neutral surface layer the second non-dimensional quantity is in principle obtained by normalizing one of the three quantities $z$, $\tau_*$ and $F_{b*}$ by the other two. This procedure leads to $\Phi_{2\tau} = \tau_*/(z \cdot F_{b*})^{2/3} = (z/L_s)^{-2/3}$, $\Phi_{2b} = F_{b*}/(z^{-1} \cdot \tau_*{}^{3/2}) = z/L_s$ and $\Phi_{2l} = z/(\tau_*{}^{3/2} \cdot F_{b*}^{-1}) = z/L_s$. The length scale $L_s = \tau_*{}^{3/2} F_{b*}^{-1}$ is the Monin-Obukhov stability parameter. It appears most practical to apply $\Phi_{2b} = \Phi_{2l} = z/L_s = \zeta_s$ as parameter $\Pi_2$. Thus we obtain the flux-profile gradient relationship

$$\frac{kz}{\tau_*{}^{1/2}} \frac{\partial U}{\partial z} = \Phi_U(\zeta_s) = k\Phi_{U0} \tag{2}$$

Similarly the flux-profile gradient relationships for potential temperature ($\theta$) and specific humidity ($q$) become

$$\frac{k_\theta z {\tau_*}^{1/2}}{F_{\theta *}} \frac{\partial \theta}{\partial z} = \Phi_\Theta(\zeta_s) = k_\theta \Phi_{\Theta 0}, \tag{3}$$

$$\frac{k_q z {\tau_*}^{1/2}}{F_{q *}} \frac{\partial q}{\partial z} = \Phi_Q(\zeta_s) = k_q \Phi_{Q 0}. \tag{4}$$

Note that the multiplication with $k$, $k_\theta$ and $k_q$ in (2) (3) and (4), respectively, yields $\Phi_M(0) = \Phi_T(0) = \Phi_Q(0) = 1$.

## Generalized similarity for the neutral and stable boundary layer

The flux-profile gradient relations above are based on a number of assumptions, including the assumption of constant turbulence fluxes equal to their surface values. In Numerical Weather Prediction (NWP) models the lowest model level is typically a few tenths of meters above the surface. The stable boundary layer (SBL) and not least its surface layer may therefore become unresolved by NWP models if the stability, e.g. in terms of $\zeta_s$, becomes sufficiently large. In such situations the depth of the SBL becomes comparable to or even smaller than the depth of the lowest model layer in most NWP models. Consequently, NWP models must be capable of parameterizing turbulence fluxes that varies with height in the SBL. Equation (2) to (4) can be generalized to include the effect of height dependent turbulence fluxes in the SBL as well as the neutral boundary layer (NBL) by utilizing the local height dependent fluxes as scaling parameters in place of the surface values $\tau_*$ and $F_{b*}$. Equation (2) to (4) then reads

$$\frac{kz}{\tau^{1/2}} \frac{\partial U}{\partial z} = \Phi_U(\zeta) \tag{5}$$

$$\frac{k_\theta z \tau^{1/2}}{F_\theta} \frac{\partial \theta}{\partial z} = \Phi_\Theta(\zeta) \tag{6}$$

$$\frac{k_q z \tau^{1/2}}{F_q} \frac{\partial q}{\partial z} = \Phi_Q(\zeta) \tag{7}$$

In these equations $\tau$, $F_\theta$ and $F_q$ are local (i.e. height dependent) turbulent fluxes of kinematic momentum, heat and moisture, respectively. The local stability parameter is defined $\zeta = z/L$, where $L = \tau^{3/2}/F_b$ is the local Monin-Obukhov length scale. In cases with a shallow SBL wind at the lowest level in NWP models may be in near gradient or geostrophic wind balance, which means that the SBL including its surface layer is influenced by the rotation of the Earth as pointed out by e.g. Zilitinkevich and Esau (2005). In the same paper they also showed that static stability in the air above the SBL or NBL has a non-local effect on the turbulence in the boundary layer. A generalization, taking into account the effect of the rotation of the Earth and the non-local effect of the static stability in the atmospheric layer on top of the NBL/SBL adds two new quantities, the Coriolis parameter $f$ and the Brunt-Vaisala frequency $N$ in the atmosphere on top of the planetary boundary layer (PBL), to the list of dimensional parameters without adding new fundamental dimensions and formally (5) to (7) become

$$\frac{kz}{\tau^{1/2}} \frac{\partial U}{\partial z} = \Phi_U(\zeta, \zeta_f, \zeta_N) \tag{8}$$

$$\frac{k_\theta z \tau^{1/2}}{F_\theta} \frac{\partial \theta}{\partial z} = \Phi_\Theta(\zeta, \zeta_f, \zeta_N) \tag{9}$$

$$\frac{k_q z \tau^{1/2}}{F_q}\frac{\partial q}{\partial z} = \Phi_Q(\zeta, \zeta_f, \zeta_N) \tag{10}$$

In these equations $\zeta_f = z/(\tau^{1/2}|f|^{-1})$ and $\zeta_N = z/(\tau^{1/2}N^{-1})$.

It would be a considerable simplification if the three non-dimensional parameters in the $\Phi$-functions in equations (8) to (10) could be combined into one parameter. Work by Esau and Byrkjedal, (2006), Esau and Zilitinkevich (2006) and Zilitinkevich and Esau (2007) indeed shows that it is possible with reasonable accuracy to approximate the $\Phi$-functions for momentum and sensible heat by the relatively simple polynomials

$$\Phi_M = 1 + C_{U1}\zeta_*, \quad \Phi_\theta = 1 + C_{\theta 1}\zeta_* + C_{\theta 2}{\zeta_*}^2, \tag{11}$$

where $\zeta_* = z/L_*$ and $L_*$ is obtained by a linear combination of the squared reciprocals of the three length scales $L$, $L_f$ and $L_N$ such that

$$\frac{1}{L_*} = \left[\left(\frac{1}{L}\right)^2 + \left(\frac{C_N}{L_N}\right)^2 + \left(\frac{C_f}{L_f}\right)^2\right]^{1/2} \tag{12}$$

In (12) $C_N = 0.4$ and $C_f = 1$ are empirical constants. Fitting to field data from SHEBA (Uttal et al., 2002) and to numerical data from Large Eddy Simulations (LES) collected in the LES DATABASE64 (Beare et al., 2006; Esau and Zilitinkevich, 2006) resulted in the following estimates of non-dimensional constants: $C_{U1} = 2$, $C_{\theta 1} = 1.6$ and $C_{\theta 2} = 0.2$. Zilitinkevich and Esau (2005) and Esau and Byrkjedal (2006) also found quasi-universal dependencies of the fluxes of turbulence normalized by their respective surface values on the hight $z$ normalized by the PBL height $h$. They found

$$\frac{\tau}{\tau_*} = \exp\left[-\frac{8}{3}\left(\frac{z}{h}\right)^2\right], \quad \frac{F_b}{F_{b*}} = \exp\left[-2\left(\frac{z}{h}\right)^2\right], \tag{13}$$

where $F_{b*}$ is the buoyancy flux at the surface.

The equilibrium PBL height $h_E$ is mainly dependent on the parameters $N$, $|f|$, $\tau_*$ and $F_{b*}$. The latter two are the surface boundary values of $\tau$ and $F_b$. The parameters can all be considered as external to the PBL. It follows from the Buckingham Pi theorem that the functional relationship $\phi_h(h_E, \tau_*, F_{b*}, |f|, N) = 0$ takes the form $\Pi_1 = \Phi_h(\Pi_2, \Pi_3)$. One possible relationship is

$$\frac{h_E}{L_f} = \Phi_h\left(\frac{L_f}{L_N}, \frac{L_f}{L_s}\right), \tag{14}$$

where $L_f = {\tau_*}^{1/2}|f|^{-1}$, $L_N = {\tau_*}^{1/2}N^{-1}$ and $L_s = {\tau_*}^{3/2}F_{b*}^{-1}$. Using a multi-limit h-model (Zilitinkevich and Mironov 1996; Zilitinkevich and Baklanov, 2002) Zilitinkevichi, Esau and Baklanov 2006, obtained

$$\left(\frac{L_f}{h_E}\right)^2 = \frac{1}{C_R^2} + \frac{1}{C_{CN}^2}\frac{L_f}{L_N} + \frac{1}{C_{NS}}\frac{L_f}{L_s}, \tag{15}$$

where $C_R = 0.6$, $C_{CN} = 1.36$ and $C_{NS} = 0.51$ are empirical dimensionless constants.

## Equatorial boundary layer height

Equation (15) clearly does not apply at the Equator. If the Coriolis term $f \cdot \cot\phi$ does not influence the PBL structure in the Equatorial region the PBL height there ($h_{eq}$) might be expected only to be determined by $\tau_*$, $F_{b*}$ and $N$, yielding

$$\frac{h_{Eq}}{L_N} = \Phi_{hq}\left(\frac{L_N}{L_s}\right), \tag{16}$$

or if an equation similar to (15) is assumed

$$\left(\frac{L_N}{h_{eq}}\right)^2 = \frac{1}{c_{e1}^2} + \frac{1}{c_{e2}^2}\frac{L_N}{L_s} \qquad \text{or} \qquad h_{eq} = c_{e1}\frac{\tau_*^{1/2}}{N}\left(1 + \frac{c_{e1}^2 F_{b*}}{c_{e2}^2 N \tau_*}\right)^{-1/2}, \tag{17}$$

where $c_{e1}$ and $c_{e2}$ are non-dimensional constants. In principle one could add for example $\cos\phi \cdot L_f^2 \cdot (L_N/h_{eq})^2$ to the right hand side of (15). However, in view of the uncertainties about the values of $c_{e1}$ and $c_{e2}$ and possible non-existence of an equilibrium height in the equatorial region we take here a pragmatic and simpler solution by adding $L_f^2/h_T^2$ to the rhs. of (15) and assuming that $h_T$ is constant.

## Iterative solution for local fluxes

Vertical integration of (2) to (4) leads to

$$U(z) - U(z_0) = \frac{\tau_*{}^{1/2}}{k}\left[\ln\frac{z}{z_0} + \psi_{U_0}\left(\frac{z}{L_0}, \frac{z_0}{L_0}\right)\right] \tag{18}$$

$$\theta(z) - \theta(z_0) = \frac{F_{\theta*}}{k_\theta \tau_*{}^{1/2}}\left[\ln\frac{z}{z_0} + \psi_{\Theta_0}\left(\frac{z}{L_0}, \frac{z_0}{L_0}\right)\right] \tag{19}$$

$$q(z) - q(z_0) = \frac{F_{q*}}{k_q \tau_*{}^{1/2}}\left[\ln\frac{z}{z_0} + \psi_{Q_0}\left(\frac{z}{L_0}, \frac{z_0}{L_0}\right)\right] \tag{20}$$

These are mean profile functions valid in an atmospheric surface layer satisfying the assumptions made in classical Monin-Obukhov similarity. The key assumption is that the kinematic turbulent fluxes of momentum, heat and moisture in the surface layer can be treated as constants, equal to their values at the surface. The $\Psi$-functions are obtained by vertical integration of the empirical determined non-dimensional profile-gradient $\Phi$-functions in (2) to (4). By definition $U(z_0) = 0$, where $z_0$ is the roughness length for momentum. In general $\theta(z_0)$ and $q(z_0)$ deviate from their surface values $\theta_s$ and $q_s$. They are related by

$$\theta_s(z_{0\theta}) = \theta(z_0) + \frac{F_{\theta*}}{k_\theta \tau_*{}^{1/2}}\ln\frac{z_0}{z_{0\theta}} \tag{21}$$

$$q_s(z_{0q}) = q(z_0) + \frac{F_{q*}}{k_q \tau_*{}^{1/2}}\ln\frac{z_0}{z_{0q}}, \tag{22}$$

where $z_{0\theta}$ and $z_{0q}$ are roughness lengths for temperature and moisture, respectively. $\Psi$-functions similar to those in (18) to (20) can in principle be obtained from (8) to (10) in combination with (12).

However, due to their complexity such functions have been determined from the LES DATABASE64. It turned out that the functions could be reasonably accurately approximated by the simple power laws (Zilitinkevich and Esau, 2007):

$$\Psi_U = C_U \zeta_*^{5/6} \quad (23)$$
$$\Psi_\Theta = C_\theta \zeta_*^{4/5} \quad (24)$$
$$\Psi_Q = \Psi_\Theta \quad (25)$$

with dimensionless constants $C_U = 3.0$ and $C_\theta = 2.5$. Using (23) to (25) in (18) to (20) gives

$$\frac{kU}{\tau^{1/2}} = \ln\frac{z}{z_0} + C_U \left(\frac{z}{L}\right)^{5/6} \left[1 + \left(\frac{L}{L_e}\right)^{4/5}\right]^{5/12} \quad (26)$$
$$\frac{k_\theta \tau}{-F_\theta}\left(\theta(z) - \theta_0\right) = \ln\frac{z}{z_0} + C_\Theta \left(\frac{z}{L}\right)^{4/5} \left[1 + \left(\frac{L}{L_e}\right)^{2}\right]^{2/5} \quad (27)$$
$$\frac{k_q \tau}{-F_q}\left(q(z) - q_0\right) = \ln\frac{z}{z_0} + C_Q \left(\frac{z}{L}\right)^{4/5} \left[1 + \left(\frac{L}{L_e}\right)^{2}\right]^{2/5}, \quad (28)$$

where $L_e^2 = \frac{\tau}{(C_N N)^2 + (C_f f)^2}$, $\theta_0 = \theta(z_0)$ and $q_0 = q(z_0)$. If $C_Q = C_\Theta$ and $k_q = k_\theta$ the last two equations can be combined to

$$\frac{k_\theta \tau}{-F_b}\left(b - b_0\right) = \ln\frac{z}{z_0} + C_\Theta \left(\frac{z}{L}\right)^{4/5} \left[1 + \left(\frac{L}{L_e}\right)^{2}\right]^{2/5}, \quad (29)$$

where $b = \beta\theta_v$ is the buoyancy acceleration and $F_b = \beta F_\theta + 0.61 g F_q$ is the buoyancy flux.

For given values of $f$ and $N$ equation (26) and (29) must be solved iteratively for the local stress $\tau(z)$ and the local buoyancy flux $F_b(z)$. Thereafter the sensible and latent heat flux are obtained from

$$F_\theta = F_b \Big(\beta + 061 g \frac{q - q_0}{\theta - \theta_0}\Big)^{-1} \quad (30)$$
$$F_q = F_\theta \frac{q - q_0}{\theta - \theta_0} \quad (31)$$

The iterative solution of (26) and (29) is obtained by relatively few iterations for neutral and weakly to moderately stable boundary layers, but for large stability ($z/L > 1$) the convergence becomes increasingly slow. To obtain fast convergence over the entire range $z/L \geq 0$ the first guess on $\tau$ and $F_b$ is based on asymptotic values of these fluxes multiplied by functions depending on the bulk Richardson number $Ri$ defined by $Ri = z\frac{b - b_0}{U^2}$, where $b$ and $U$ are values of buoyancy acceleration and wind speed at the lowest model level. For large $z/L$ (26) and (29) can be approximated by

$$\frac{kU}{\tau^{1/2}} \approx c_U \left(\frac{z}{L}\right)^{5/6} \quad (32)$$

and

$$\frac{k_\theta \tau^{1/2}\left(b-b_0\right)}{F_b} \approx c_\theta \left(\frac{z}{L}\right)^{4/5} \tag{33}$$

From the latter two equations follow

$$z/L_{lim} \approx \left(\frac{C_U^2 k_\theta}{C_\theta k^2} Ri\right)^{15/2} \tag{34}$$

The asymptotic $z/L_{lim}$ are used in (26) and (29) to calculate corresponding asymptotic values $\tau_{lim}$ and $F_{blim}$. The first guess values in the iteration is then $\tau = \Gamma_\tau(Ri)\tau_{lim}$ and $F_b = \Gamma_{Fb}(Ri)F_{blim}$, where $\Gamma_\tau = \Gamma_{\tau_1} + \Gamma_{\tau_2}$ and $\Gamma_{Fb} = \Gamma_{Fb1} + \Gamma_{Fb2}$ are empirical functions of $Ri$ determined by

$$\Gamma_{\tau_1} = \max\{C_a(1-\frac{Ri}{Ri_c})^2(1-\frac{Ri}{Ri_1}), C_b\} \tag{35}$$

$$\Gamma_{\tau_2} = \min\{\max\{C_c\left(\frac{Ri-Ri_1}{Ri_c-Ri_1}\right)\exp^{-1}(Ri_c-Ri), 0\}, 1\} \tag{36}$$

$$\Gamma_{F_{b1}} = \max\{\Gamma_{\tau_1}(1-\sqrt{2}Ri)^{-1}, C_d\} \tag{37}$$

$$\Gamma_{F_{b2}} = \Gamma_{\tau_2} \tag{38}$$

The constants have the values $Ri_c = 0.25$, $Ri_1 = 0.4Ri_c$, $C_a = 0.95$, $C_b = 0.1793$, $C_c = 0.96$ and $C_d = 0.2$. The first guess values calculated by the procedure described above give very fast convergence in the entire parameter space. The result after only 1 iteration is practically identical with the result after 50 iterations. The convergence in the second iteration method involving surface fluxes and PBL height is a little slower. This method is described in the next section.

## Iterative solution for boundary layer height and surface fluxes

Once the local fluxes have been calculated the surface fluxes and equilibrium PBL height are calculated iteratively from (13) and (15), the latter modified to

$$\left(\frac{L_f}{h_E}\right)^2 = \frac{1}{C_R^2} + \frac{1}{C_{CN}^2}\frac{L_f}{L_N} + \frac{1}{C_{NS}}\frac{L_f}{L_s} + w(\phi)\left(\frac{L_f}{h_T}\right)^2, \tag{39}$$

to avoid an infinite PBL height at the Equator. Influence of the tropical PBL height at mid- and high latitudes can be eliminated by multiplying the last term on the rhs. of (39) by a weighting function, for example $w(\phi) = \max(0, 1 - f/f_0)$, where $f_0 = 10^{-4}\,s^{-1}$.

The iteration begins with calculation of a preliminary PBL height $h_{ini}$ from (39) with $L_N$, $L_s$ and $L_f$ replaced by their local values, i.e. $L_{Nl} = \tau^{1/2}N^{-1}$, $L_{sl} = \tau^{3/2}F_b^{-1}$ and $L_{fl} = \tau^{1/2}|f|^{-1}$ with the local $\tau$ and $F_b$ obtained from the iteration described in the previous section. A preliminary surface stress $\tau_{*ini}$ is next calculated from (13), i.e. $\tau_{*ini} = \tau \cdot [\exp\left(-8/3 \cdot (z/h_{ini})\right)]^{-1}$. It follows from (13) and (39) that if $\tau_{*ini}$ was the correct surface stress we would have

$$G = K_N^2 + K_f \cdot \frac{F_{b*ini}}{\tau_{*ini}} + w(\phi)\frac{\tau_{*ini}}{h_T^2} - \frac{3}{8}z^{-2}\tau_{*ini}\ln\frac{\tau_{*ini}}{\tau} = 0, \tag{40}$$

where $K_N = (f^2 \cdot C_R^{-2} + N|f|C_{CN}^{-2})^{1/2}$ and $K_f = |f|C_{NS}^{-2}$. The square frequency function $G$ is only a function of $\tau_*$ since it follows from (13) that

$$|F_{b*}| = |F_b|\left(\frac{\tau_*}{\tau}\right)^{3/4} \tag{41}$$

Therefore (40) can be written

$$G(\tau_*) = K_N^2 + K_f\frac{|F_b|}{\tau}\left(\frac{\tau}{\tau_*}\right)^{1/4} + w(\phi)\frac{\tau_*}{h_T^2} - \frac{3}{8}z^{-2}\tau_*\ln\frac{\tau_*}{\tau}. \tag{42}$$

In the iteration a new value $\tau_* = \tau_* + \delta\tau_*$ is determined such that $G_l(\tau_*) = 0$, where $G_l$ is a linearization of $G$. As long as $\delta\tau_*/\tau_* << 1$ we have

$$G \approx G_l = G(\tau_*) + \left(\frac{1}{4}K_f\frac{|F_b|}{\tau}\left(\frac{\tau}{\tau_*}\right)^{1/4}\frac{1}{\tau_*} + w(\phi)\frac{\tau_*}{h_T^2} + \frac{3}{8}z^{-2}\tau_*(1+\ln\frac{\tau_*}{\tau})\right)\delta\tau_* = 0 \tag{43}$$

or

$$\delta\tau_* = \frac{G(\tau_*)}{\left(\frac{1}{4}K_f\frac{|F_b|}{\tau}\left(\frac{\tau}{\tau_*}\right)^{1/4}\frac{1}{\tau_*} + w(\phi)\frac{\tau_*}{h_T^2} + \frac{3}{8}z^{-2}(1+\ln\frac{\tau_*}{\tau})\right)}. \tag{44}$$

In the first iteration $\tau_* = \tau_{*ini}$. In a very stable (and shallow) PBL the ratio $\tau/\tau_*$ may become very small. In such cases the first guess values $h_{ini}$ and $\tau_{*ini}$ deviate considerably from their final values. To obtain fast convergence in these cases $\tau_{*ini}$ in (40) is replaced by
$\tau_{*ini} = kU(1 + 10Ri)^3(\ln z/z_0)^{-1}$.

Once $\tau_*$ has been calculated by iteration from (43) and (44), $F_{b*}$ and $h_E$ are obtained from

$$|F_{b*}| = |F_b|\left(\frac{\tau_*}{\tau}\right)^{3/4} \tag{45}$$

and

$$h_E = \left[\frac{\tau_*}{K_N^2 + K_f\frac{|F_{b*}|}{\tau_*} + w(\phi)\frac{\tau_*}{h_T^2}}\right]^{1/2} \tag{46}$$

Figure 1 and 2 show that rapid convergence is obtained in the tested parameter space including the mid-latitude and equatorial neutral and stably stratified PBL both over a rough and smooth surface and with a large and small free flow static stability. The result after two iterations is practically identical with that obtained after fifty iterations.

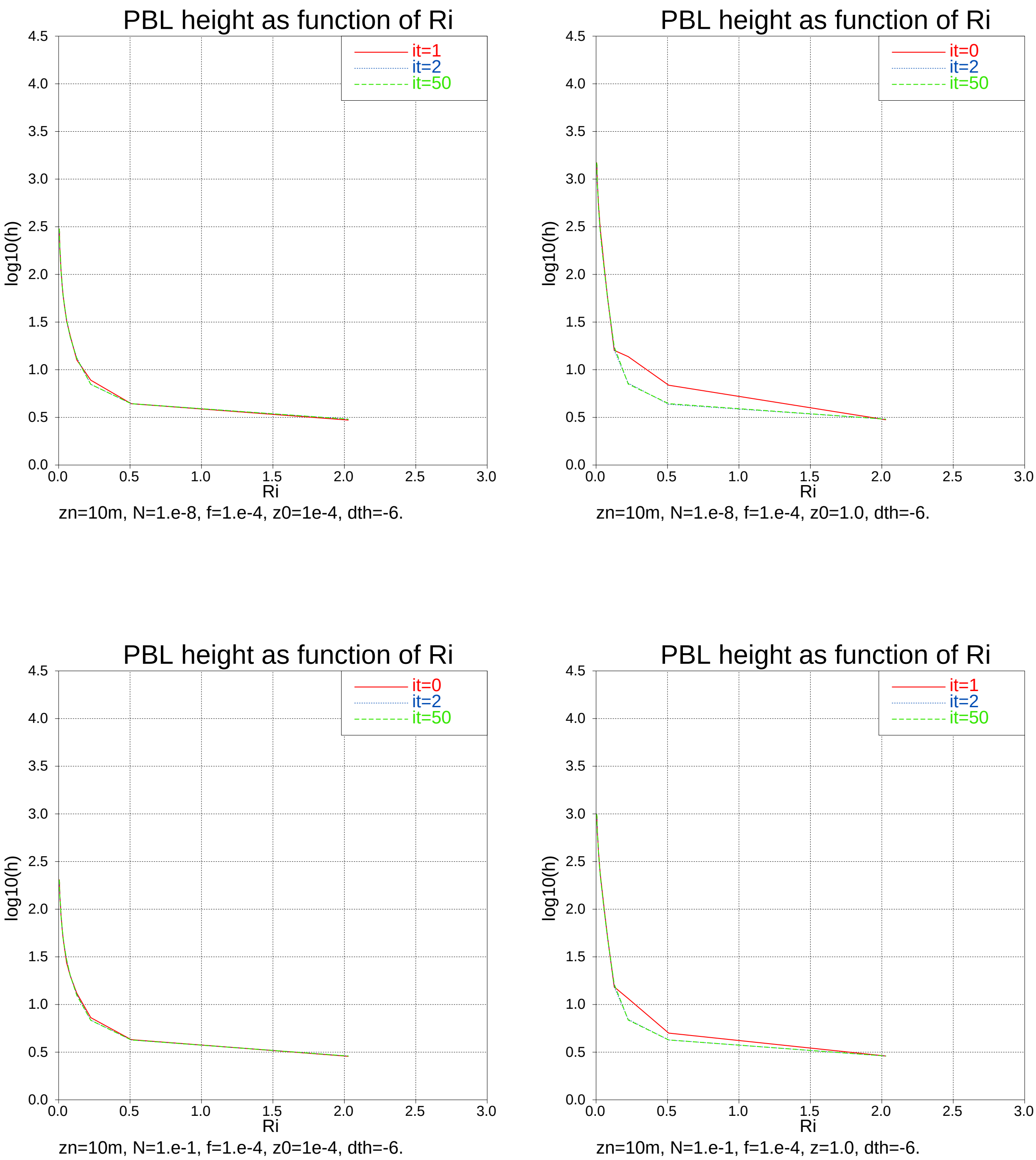


**Figure 1**: Iterative solution for the equilibrium PBL height at mid-latitudes as function of the bulk Richardson number $Ri$, calculated from wind and temperature at 10 m height ($zm = 10\,m$). Upper row is for a weak free flow static stability ($N = 1.e^{-8}\,s^{-1}$) and from left to right for a smooth ($z_0 = 1.e^{-4}$) and rough surface ($z_0 = 1.0$), respectively. Lower row is like the top row, but with a strong free flow static stability ($N = 0.1\,s^{-1}$). Number of iterations applied is given by $it$.

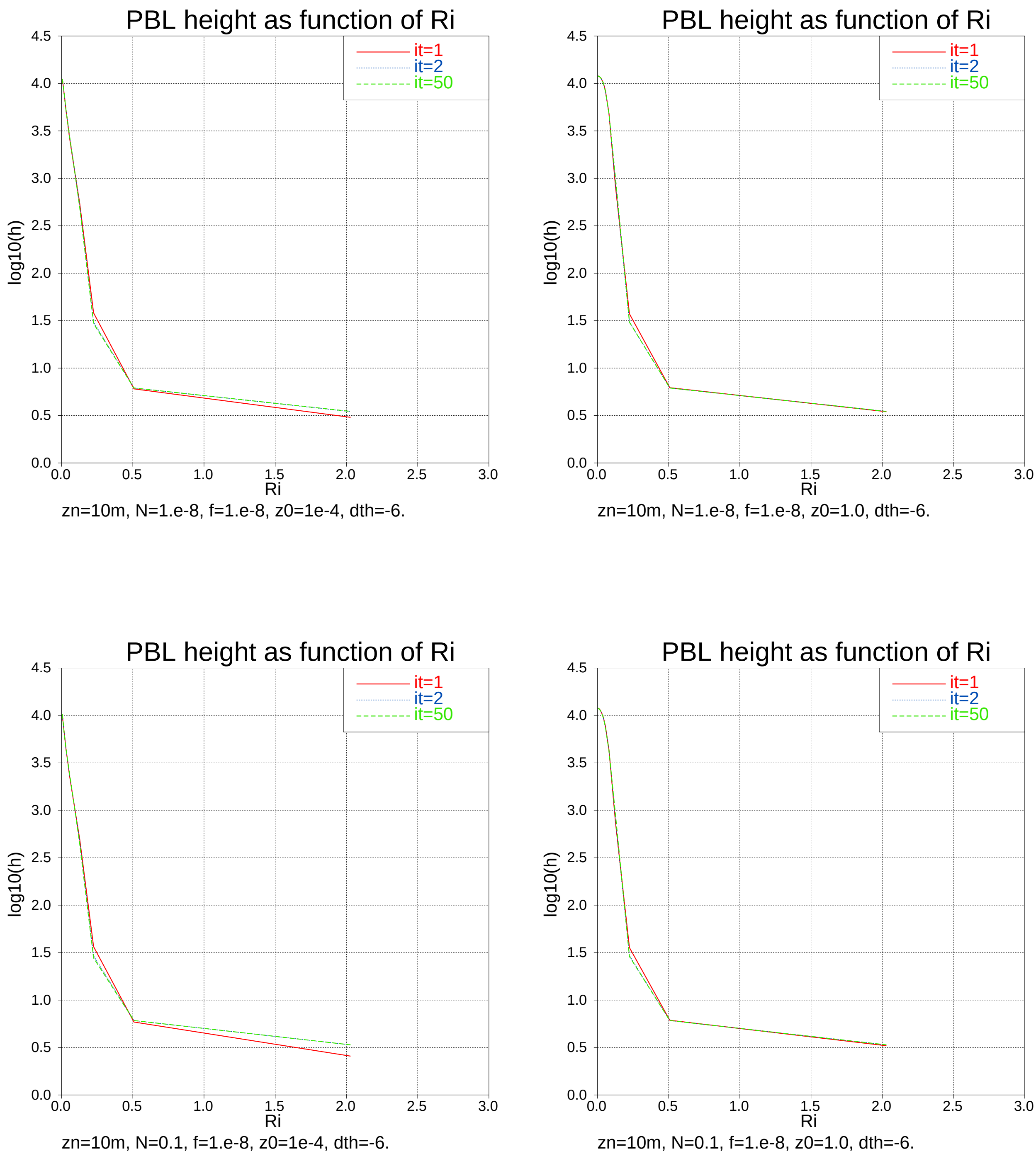


**Figure 2**: Iterative solution for the equilibrium PBL height in the equatorial PBL as function of the bulk Richardson number $Ri$b, calculated from wind and temperature at 10 m height ($zm = 10\,m$). Upper row is for a weak free flow static stability ($N = 1.e^{-8}\,s^{-1}$) and from left to right for a smooth ($z_0 = 1.e^{-4}$) and rough surface ($z_0 = 1.0$), respectively. Lower row is like the top row, but with a strong free flow static stability ($N = 0.1\,s^{-1}$). Number of iterations applied is given by $it$.

## Comments and concluding remarks

In NWP models like HIRLAM the angle $\alpha(z)$ between the stress $\vec{\tau}(z)$ and the surface stress $\vec{\tau}_*$ must be calculated. An equation for the surface cross isobar angle $\alpha_*$ is presented in Zilitinkevich and

Esau, 2007. However, it appears as if the latter equation does not work properly in some cases. It is suggested here to use a slightly modified form of equation (10) in Esau and Zilitinkevich, 2006. The modified form reads

$$\sin\alpha(z) = \left(\frac{\tau}{\tau_*} - 1\right) C_\alpha k (\ln Ro_m + C_{*m})^{-1} \left(C_{N3}{\mu_N}^{3/4} + 1\right) \cdot \left((C_{S3}\mu_s)^3 \cdot (1 + \mu_N \cdot w(\phi))^{-1} + 1\right). \tag{47}$$

In (47) $Ro_m = V_h \cdot (fz_0)^{-1} + 1$ is a kind of modified surface Rossby number, $V_h$ is the wind speed at the top of the PBL, $C_{*m} = C_* \left(1 - [(0.4 \cdot \ln Ro_m)^3 + 1]^{-1}\right)$, $C_* = -4.2$, $C_\alpha = 4$, $C_{N3} = 0.03$, $C_{S3} = 0.0012$, $\mu_N = N \cdot f^{-1}$ and $\mu_s = (F_{b*}(f^3 z_0)^{-1})^{1/3}$. The modifications have been done to obtain a formula that works at low latitudes, at low surface Rossby numbers and gives the angle between the the stress vector at hight $z$ and the surface stress vector, i.e. an angle varying with height, by definition identical zero at the surface and equal to the surface cross isobar angle for $z = h_E$.

The pilot study presented in the present report shows that from a computational economy point of view it is feasible to implement the generalized similarity parameterization for the neutral and stable PBL in the HIRLAM model. The pilot study applies an equilibrium PBL height $h_E$. In the implementation in the HIRLAM model it might turn out to be an advantage to replace $h_E$ with a PBL height determined by a prognostic equation for this quantity. In the latter case the implementation in the HIRLAM NWP model becomes considerably more involving. Therefore it is suggested as the next step to implement the scheme in its simplest form, as it is presented in the present pilot study, into a column version of HIRLAM. If the column tests turns out to be satisfactory the scheme is then ready for implementation and final tests in the HIRLAM NWP model.

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